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New Boundary Conformal Field Theories Indexed by the Simply-Laced Lie Algebras

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Abstract

We consider the field theory of N massless bosons which are free except for an interaction localized on the boundary of their 1+1 dimensional world. The boundary action is the sum of two pieces: a periodic potential and a coupling to a uniform abelian gauge field. Such models arise in open string theory and dissipative quantum mechanics, and possibly in edge state tunneling in the fractional quantized Hall effect. We explicitly show that conformal invariance is unbroken for certain special choices of the gauge field and the periodic potential. These special cases are naturally indexed by semi-simple, simply laced Lie algebras. For each such algebra, we have a discrete series of conformally invariant theories where the potential and gauge field are conveniently given in terms of the weight lattice of the algebra. We compute the exact boundary state for these theories, which explicitly shows the group structure. The partition function and correlation functions are easily computed using the boundary state result.

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1. Introduction

Free field theory on a 1+1 dimensional manifold with boundary has been the focus of a number of recent works. With the proper choice of boundary interactions, such theories can describe dissipative quantum mechanics [1,2], tunneling between quantum Hall edges [3], impurity scattering [4], open strings in background fields [5,6] , and monopole catalysis [7,8,9]. Typically, the interesting physics lies mostly in the infrared limit of these theories, and so it is of considerable interest to identify the conformally-invariant fixed points to which they flow.

We will study the following Euclidean action:

$$S = S_{bulk} + S_{gauge} + S_{pot}, \tag{1.1a}$$

$$S_{bulk} = \frac{1}{8\pi} \int_0^T d\tau \int_0^l d\sigma \Big(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X \Big), \tag{1.1b}$$

$$S_{gauge} = \frac{i}{8\pi} \int_0^T d\tau \, X(0,\tau) \cdot B \cdot \partial_\tau X(0,\tau), \tag{1.1c}$$

$$S_{pot} = \int_0^T d\tau \, V(X(0,\tau)), \tag{1.1d}$$

where

$$V(X) = \sum_k \Big(V_{\omega_k} e^{i\omega_k \cdot X} + V_{-\omega_k} e^{-i\omega_k \cdot X} \Big).$$

$X(\sigma, \tau)$ is a real-valued N component bosonic field, B is a real anti-symmetric matrix, and the ω_k are real N component vectors. The complex parameters V_ω obey the relation $V_{-\omega} = \bar{V}_\omega$, so that $V(X)$ is a real potential. S_{bulk} describes a free massless theory on the interval $0 \leq \sigma \leq l$. The terms S_{gauge} and S_{pot} are perturbations localized at the $\sigma = 0$ boundary. In principle, we could also add similar perturbations localized at $\sigma = l$, but we will omit such terms since our later analysis shows that we can treat each boundary independently.

From the point of view of string theory, this action describes a bosonic open string whose endpoints feel the presence of a background gauge field and periodic potential. The conformal fixed points of the action correspond to background field configurations which solve the classical open string field equations.

Our action can also be used to study the dissipative quantum mechanics (DQM) of a non-relativistic particle subject to the periodic potential $V(X)$ and magnetic field B (the Wannier-Azbel-Hofstadter model [10]). The correspondence between open string theory

and DQM has been outlined in detail [2,11], and may be summarized as follows: the endpoint of the string corresponds to the location of the DQM particle. The fields felt by the string endpoint correspond to the fields felt by the particle. As the string endpoint moves, it loses energy by exciting the modes of the string, which corresponds to the DQM particle losing energy by exciting a dissipative bath of oscillators. The mapping between the open string and DQM becomes exact in the infrared limit, and so it is of greatest interest to study the conformal fixed points of the action: at these points, the correlators of X show scaling behaviour, indicating that we are at a transition point between localized and delocalized behavior for the DQM particle.

Our action S subsumes and generalizes some special cases considered in a number different papers. The oldest preceding paper [6] studies the quadratic action $S_{bulk} + S_{gauge}$, i.e., the $V(X) = 0$ limit of our model. The exact solution of the $V = 0$ case shows that the action is conformally invariant for generic values of B. In [12], the authors considered the simplest $V \neq 0$ case: $N = 1$, $B = 0$, and $V(X) = V_0 \cos(X/\sqrt{2})$. That action proves to be conformally invariant for any value of V_0, thanks to the presence of an $SU(2)_1$ Kac-Moody symmetry. A number of papers [11,13,14] have been devoted to the next simplest case, with $N = 2$, $B_{ij} = b\epsilon_{ij}$, and $V(X) = V_0(\cos(aX) + \cos(aY))$. In this case, the analysis depends in a critical way on the parameter b. If b is an integer, the theory is conformally invariant for any value of V_0, provided we set $a = \sqrt{\frac{1+b^2}{2}}$. In direct analogy with the $N = 1$ case, an $SU(2)_1 \times SU(2)_1$ symmetry appears in the solution. When b is not an integer, things are not so simple: if we set $a = \sqrt{\frac{1+b^2}{2}}$, then $V_0 = 0$ is the only fixed point. If we take $a < \sqrt{\frac{1+b^2}{2}}$, then perturbative calculations indicate that $V_0 = 0$ becomes unstable, and V_0 flows to a finite fixed point value in the infrared. No Kac-Moody algebra arises for non-integer values of b, and consequently no exact results are available.

In this paper, we seek to generalize the results obtained for integer b in the $N = 2$ model. Our approach is to choose fixed values for B and ω_k and turn on $V(X)$ as a perturbation. Using boundary state technology, we will show that there is a series of special values for B and ω_k such that the perturbation $V(X)$ is exactly marginal. In these special cases, one may exactly solve the theory due to the presence of a level 1, simply laced affine Kac-Moody algebra. Furthermore, for any semi-simple, simply laced Lie algebra g, there exists a discrete series of choices for B and ω_k that guarantee conformal invariance. The constants V_{ω_k} parameterize a manifold of fixed points isomorphic to the corresponding Lie group G. Interestingly, $B = 0$ is allowed only if g is a direct sum of $su(2)$ algebras; the gauge field interaction is indispensable for realizing all the other algebras. We will

compute the exact boundary state for these theories, from which we may easily compute the partition function and correlation functions.

2. Basic Setup

Our primary goal is to compute the functional integral on a cylinder of length l and circumference T. As is well known [15], there are two equivalent pictures for the path integral. In the first picture, we are computing the partition function $Z = tr(e^{-TH_{open}})$ for an open string of length l at temperature T; the Hamiltonian H_{open} is that of an open string with boundary action described by $S_{gauge} + S_{pot}$. In the dual picture, we are computing the amplitude $Z = \langle N|e^{-lH_{closed}}|V\rangle$, which describes a free closed string of length T propagating for time l between two 'boundary states'. The boundary states correspond to the ends of the cylinder, and they conveniently summarize the dynamics present at the boundaries of the open string. In this case, $|N\rangle$ represents the free Neumann boundary condition and $|V\rangle$ represents the boundary condition induced by $S_{gauge} + S_{pot}$. Conformal invariance of the open string action translates into the Virasoro constraint $(L_n - \bar{L}_{-n})|V\rangle = 0$.

In this paper, we will work in the closed string channel and focus on computing the boundary state $|V\rangle$. Consequently, we can use the standard closed string chiral mode expansions:

$$X(z, \bar{z}) = X(z) + \bar{X}(\bar{z}),$$

$$X(z) = q - ip\ln(z) + \sum_{n\neq 0} \frac{i}{n}\alpha_n z^{-n},$$

$$\bar{X}(\bar{z}) = \bar{q} - i\bar{p}\ln(\bar{z}) + \sum_{n\neq 0} \frac{i}{n}\bar{\alpha}_n \bar{z}^{-n},$$

where

$$[q_j, p_k] = [\bar{q}_j, \bar{p}_k] = \quad i\delta_{j,k},$$

$$[q_j, q_k] = [p_j, p_k] = [\bar{q}_j, \bar{q}_k] = [\bar{p}_j, \bar{p}_k] = [q_j, \bar{p}_k] = [\bar{q}_j, p_k] = 0,$$

$$[\alpha_{nj}, \alpha_{mk}] = [\bar{\alpha}_{nj}, \bar{\alpha}_{mk}] = \quad n\delta_{j,k}\delta_{n+m,0},$$

$$[\alpha_{nj}, \bar{\alpha}_{mk}] = 0,$$

$$\alpha_{nj}^{\dagger} = \alpha_{-nj}, \qquad \alpha_{0i} = p_i$$

$$z = e^{2\pi(\tau+i\sigma)/T} \qquad \text{and} \qquad \bar{z} = e^{2\pi(\tau-i\sigma)/T}.$$

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The Hamiltonian is $H_{closed} = \frac{2\pi}{T}(L_0 + \bar{L}_0)$, where we have the standard Virasoro algebra

$$L_0 = \frac{1}{2}p^2 + \sum_{n=1}^{\infty} \alpha_n^\dagger \cdot \alpha_n - \frac{N}{24},$$

$$L_k = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_n \cdot \alpha_{k-n} :,$$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{N}{12}(n^3 - n)\delta_{n+m,0},$$

with corresponding expressions for $\bar{L}_n$.

Since $[X(z), \bar{X}(\bar{w})] = 0$, we can generate the closed string Hilbert space by taking tensor products of states built from left-moving and right-moving modes. There is one caveat: we must require $X(z) + \bar{X}(\bar{z})$ to be single-valued as $\sigma \to \sigma + T$. If X is non-compact, then only states with $p = \bar{p}$ are allowed.

We can also take $X(z, \bar{z})$ to be compactified on a lattice Λ, corresponding to the identification $X + \bar{X} \equiv X + \bar{X} + \lambda$, where $\lambda \in \Lambda$. Our Hilbert space is then subject to the restriction $p - \bar{p} \in \frac{1}{2\pi}\Lambda$. Note that we regard this as a restriction on the Hilbert space, not on the operators p and $\bar{p}$. Compactifying X also discretizes the momentum spectrum, giving us the additional restriction $p + \bar{p} \in 4\pi\Lambda^*$, where Λ^* is the dual of Λ. A summary of lattice facts and conventions is given in the appendix.

We will be interested in both the compact and non-compact cases. To avoid writing two copies of all future results, we make the simple observation that the non-compact case is actually subsumed in the compact case if we allow the degenerate lattice $\Lambda = \{0\}$; the dual lattice in that case is $\Lambda^* = \mathbb{R}^N$, so the momentum spectrum is indeed continuous.

3. Neumann, Dirichlet, and Gauge Field Boundary States

Before tackling the computation of the boundary state $|V\rangle$, it will be helpful to consider some simpler cases: Neumann, Dirichlet, and gauge field boundary states. Once we have the gauge field state $|B\rangle$, we can express $|V\rangle$ as a perturbation of $|B\rangle$.

We will begin by considering just the action S_{bulk}, which leads to Neumann boundary conditions on X in the open string channel. In the closed string channel, this means that the boundary state must satisfy $\partial_\tau X(\sigma, \tau)|N\rangle = 0$ at $\tau = 0$. In terms of oscillator modes, we have

$$(p + \bar{p})|N\rangle = (\alpha_n + \bar{\alpha}_{-n})|N\rangle = 0. \tag{3.1}$$

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This condition yields the state

$$|N\rangle = 2^{-N/4}\sqrt{\text{vol}\,(\Gamma)}\,e^{-\sum_{n=1}^{\infty}\frac{1}{n}\bar{\alpha}_n^{\dagger}\cdot\alpha_n^{\dagger}}\sum_{\gamma\in\Gamma}|p=\frac{\gamma}{\sqrt{2}},\ \bar{p}=\frac{-\gamma}{\sqrt{2}}\rangle, \qquad (3.2)$$

where $\Gamma=\frac{1}{2\pi\sqrt{2}}\Lambda$, Λ is the lattice of compactification for X, and $|p=k,\bar{p}=\bar{k}\rangle$ denotes the oscillator ground state with momenta $p=k$ and $\bar{p}=\bar{k}$.

Strictly speaking, equation (3.1) does not determine the overall normalization of $|N\rangle$. It also does not determine which momenta contribute to the sum, other than restricting $p+\bar{p}=0$. One can nevertheless verify that equation (3.2) is correct by directly computing the open string partition function with Neumann boundary conditions at both ends, then comparing the result with the partition function computed using the boundary state $|N\rangle$. Explicitly, the direct open string calculation yields

$$Z_{NN} = w^{-N/24}\prod_{n=1}^{\infty}(1-w^n)^{-N}\sum_{\gamma^*\in\Gamma^*}w^{(\gamma^*)^2},\quad\text{where}\quad w=e^{-\pi T/l}, \qquad (3.3)$$

while the closed string boundary state method yields

$$Z_{NN} = \langle N|e^{-\frac{2\pi l}{T}(L_0+\bar{L}_0)}|N\rangle \qquad (3.4)$$

$$= 2^{-N/2}\,\text{vol}\,(\Gamma)\tilde{w}^{-N/12}\prod_{n=1}^{\infty}(1-\tilde{w}^{2n})^{-N}\sum_{\gamma\in\Gamma}\tilde{w}^{\gamma^2/2},\quad\text{where}\quad \tilde{w}=e^{-2\pi l/T}.$$

Poisson resummation establishes that (3.3) and (3.4) are equivalent.

Similar considerations lead to the Dirichlet boundary equations

$$(p-\bar{p})|D\rangle = (\alpha_n-\bar{\alpha}_{-n})|D\rangle = 0 \qquad (3.5)$$

and the Dirichlet boundary state

$$|D\rangle = 2^{-N/4}\sqrt{\text{vol}\,(\Gamma^*)}\,e^{\sum_{n=1}^{\infty}\frac{1}{n}\bar{\alpha}_n^{\dagger}\cdot\alpha_n^{\dagger}}\sum_{\gamma^*\in\Gamma^*}|p=\frac{\gamma^*}{\sqrt{2}},\ \bar{p}=\frac{\gamma^*}{\sqrt{2}}\rangle.$$

Given equations (3.1) and (3.5), one may easily verify the conformal invariance constraints $(L_n-\bar{L}_{-n})|N\rangle = (L_n-\bar{L}_{-n})|D\rangle = 0$.

Finally, we will consider the action $S_{bulk}+S_{gauge}$, which results in the gauge field boundary state $|B\rangle$. In a compactified theory, we must consider what happens to S_{gauge} when we make the physically unobservable shift $X\to X+\lambda$, with $\lambda\in\Lambda$. Looking at

equation (1.1c), we see that $S_{gauge} \to S_{gauge} + \frac{i}{8\pi}\lambda' \cdot B \cdot \lambda$, where $\lambda, \lambda' \in \Lambda$. The path integral will be invariant under such shifts, provided that

$$\frac{1}{2}B\Gamma \subset \Gamma^*. \tag{3.7}$$

Varying the action $S_{bulk} + S_{gauge}$ gives a linear constraint on X at the boundary, which translates into the closed string condition $((1+B) \cdot \alpha_n + (-1+B) \cdot \bar{\alpha}_n^\dagger)|B\rangle = 0$. This leads to the solution

$$\begin{aligned}
|B\rangle =\ & 2^{-N/4}\sqrt{\det(1+B)\,\mathrm{vol}\,(\Gamma)} \\
& e^{-\sum_{n=1}^\infty \frac{1}{n}\bar{\alpha}_n^\dagger \cdot M \cdot \alpha_n^\dagger}\sum_{\gamma \in \Gamma}|p = \frac{1}{\sqrt{2}}(1-B)\gamma,\ \ \bar{p} = -\frac{1}{\sqrt{2}}(1+B)\gamma\rangle,
\end{aligned}$$

where we define the orthogonal matrix

$$M = \frac{1+B}{1-B}.$$

In the next section, we will compute $|V\rangle$ as a perturbation on $|B\rangle$. To this end, it is useful to rearrange our expression of $|B\rangle$ into the following form:

$$|B\rangle = P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R|D'\rangle. \tag{3.8}$$

We have introduced several new objects; first we have the projection operator

$$P_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega; \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Next, we define the chiral rotation and reflection operators S_B and R:

$$\begin{aligned}
S_B X(z)S_B^\dagger &= M \cdot X(z), & S_B \bar{X}(\bar{z})S_B^\dagger &= \bar{X}(\bar{z}), \\
RX(z)R^\dagger &= -X(z), & R\bar{X}(\bar{z})R^\dagger &= \bar{X}(\bar{z}), \\
S_B^\dagger S_B &= 1 \quad \text{and} \quad R = R^\dagger = R^{-1}.
\end{aligned}$$

Finally, we define a boundary state which is almost the same as the Dirichlet boundary state:

$$|D'\rangle = 2^{-N/4}\sqrt{\det(1+B)\,\mathrm{vol}\,(\Gamma)}e^{\sum_{n=1}^\infty \frac{1}{n}\bar{\alpha}_n^\dagger \cdot \alpha_n^\dagger}\sum_{\mu \in \Upsilon}|p = \mu,\ \ \bar{p} = \mu\rangle. \tag{3.10}$$

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The set Υ can be any lattice that satisfies $\frac{1}{\sqrt{2}}(1+B)\Gamma \subset \Upsilon$; the projection operator P removes any extra states. The freedom to chose Υ will be useful when we compute $|V\rangle$.

Conformal invariance of $|B\rangle$ is readily verified. From the definitions above, it is easy to see that L_n and $\bar{L}_n$ commute with the operators S_B,R, and P, so we have

$$(L_n - \bar{L}_n)|B\rangle = P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R(L_n - \bar{L}_n)|D'\rangle = 0. \tag{3.11}$$

4. The Boundary State with Gauge Field and Potential

We will now compute the boundary state $|V\rangle$ for our complete action $S = S_{bulk} + S_{gauge} + S_{pot}$. The basic idea is to regard $S_{bulk} + S_{gauge}$ as the free theory with S_{pot} as a perturbation. For S_{pot} to be well-defined in a compactified theory, we must require

$$\omega_k \in \frac{1}{\sqrt{2}}\Gamma^*. \tag{4.1}$$

Assuming this is true, we have

$$|V\rangle = e^{-\int_0^T V(X(\tau=0,\sigma))d\sigma}|B\rangle$$
$$= e^{\frac{iT}{2\pi}\oint \frac{dz}{z} V(X(z)+\bar{X}(\frac{1}{z}))} \; P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R|D'\rangle.$$

The contour of integration is the circle $|z| = 1$. Since the projection operator only depends on $p - \bar{p}$, it commutes with the potential $V(X(z) + \bar{X}(\frac{1}{z}))$. Shifting the operators around, we get

$$|V\rangle = P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R \, e^{\frac{iT}{2\pi}\oint \frac{dz}{z} V(-M^t \cdot X(z)+\bar{X}(\frac{1}{z}))}|D'\rangle. \tag{4.3}$$

Expanding the exponential, we obtain

$$|V\rangle = P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{iT}{2\pi}\right)^n \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_n}{z_n} \sum_{\nu_j \in \{\pm\omega_k\}} V_{\nu_1} \cdots V_{\nu_n}$$
$$e^{i\nu_1 \cdot (-M^t \cdot X(z_1)+\bar{X}(\frac{1}{z_1}))} \cdots e^{i\nu_n \cdot (-M^t \cdot X(z_n)+\bar{X}(\frac{1}{z_n}))}|D'\rangle. \tag{4.4}$$

4.1. Evaluation of a Typical Term

Let us focus our attention on the exponential factors in (4.4). A typical term is of the form

$$|typ\rangle = e^{i\nu_1 \cdot (-M^t \cdot X(z_1)+\bar{X}(\frac{1}{z_1}))} \cdots e^{i\nu_n \cdot (-M^t \cdot X(z_n)+\bar{X}(\frac{1}{z_n}))}|D'\rangle, \tag{4.5}$$

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where $\nu_1, \cdots \nu_n \in \{\pm\omega_k\}$. With some manipulation, we can put (4.5) in a form that will allow us to evaluate (4.4) very explicitly. All the identities that we will apply to (4.5) are easily derived from the basic Campbell-Hausdorff identities

$$e^A e^B = e^B e^A e^{[A,B]} \quad \text{and} \quad e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \tag{4.6}$$

which are true whenever $[A, [A, B]] = [B, [A, B]] = 0$.

We begin by separating the left movers from the right movers:

$$|typ\rangle = e^{-i\nu_1 \cdot M^t \cdot X(z_1)} \cdots e^{-i\nu_n \cdot M^t \cdot X(z_n)} \; e^{i\nu_1 \cdot \bar{X}(\frac{1}{z_1})} \cdots e^{i\nu_n \cdot \bar{X}(\frac{1}{z_n})}|D'\rangle. \tag{4.7}$$

Next, we note that $|D'\rangle$ converts right-movers into left-movers in the following way:

$$e^{i\nu \cdot \bar{X}(\frac{1}{z})}|D'\rangle = e^{-i\nu \cdot X(z)}|D'\rangle. \tag{4.8}$$

This identity follows from (3.5) and (4.6), providing that the lattice Υ includes the momentum ν. Since we are free to make Υ as dense as we like, we assume it contains ν.

Applying (4.8) to (4.7), we see that we can convert the rightmost right-moving exponential into a left-moving one, then commute it to the left of all the remaining right-movers. Continuing until we eliminate all the right-movers, we obtain

$$|typ\rangle = e^{-i\nu_1 \cdot M^t \cdot X(z_1)} \cdots e^{-i\nu_n \cdot M^t \cdot X(z_n)} e^{-i\nu_n \cdot X(z_n)} \cdots e^{-i\nu_1 \cdot X(z_1)}|D'\rangle. \tag{4.9}$$

Now we introduce a normal ordering convention which puts q to the left of p and $\alpha_n^\dagger$ to the left of α_n for $n > 0$. Applying (4.6), we see that

$$e^{i\nu \cdot X(z)} = (\epsilon z)^{\frac{1}{2}\nu^2} : e^{i\nu \cdot X(z)} :, \tag{4.10}$$

where ϵ is a short distance cutoff introduced to regulate the divergences due to contractions of $X(z)$ with itself. To be precise, we have regulated the short distance behaviour by redefining the canonical commutation relations:

$$[\alpha_{nj}, \alpha_{mk}] = n(1 - \epsilon)^{|n|}\delta_{j,k}\delta_{n+m,0}. \tag{4.11}$$

Note that our redefinition of the canonical commutation relations will cause a modification of (4.8). To avoid this, we make a compensating redefinition of the oscillator part of the Dirichlet state: $|D'\rangle_{oscillator} \rightarrow e^{\sum_{n=1}^{\infty} \frac{1}{n}(1-\epsilon)^n \bar{\alpha}_n^\dagger \cdot \alpha_n^\dagger}|0\rangle$.

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Our typical term is now

$$|typ\rangle = \epsilon^{\nu_1^2 + \cdots \nu_n^2} z_1^{\nu_1^2} \cdots z_n^{\nu_n^2}$$

$$: e^{-i\nu_1 \cdot M^t \cdot X(z_1)} : \cdots : e^{-i\nu_n \cdot M^t \cdot X(z_n)} : \; : e^{-i\nu_n \cdot X(z_n)} : \cdots : e^{-i\nu_1 \cdot X(z_1)} : |D'\rangle.$$

The next step is to shuffle the exponentials so that exponentials which depend on the same coordinate are next to each other. By careful application of (4.6), one may show that

$$: e^{i\nu.X(z)} :: e^{i\mu \cdot X(w)} := : e^{i\mu \cdot X(w)} :: e^{i\nu \cdot X(z)} : \quad e^{i\pi\nu \cdot \mu \, \mathrm{sgn}_\epsilon (\sigma_z - \sigma_w)}, \tag{4.13}$$

where $z = e^{2\pi i \sigma_z / T}$, $w = e^{2\pi i \sigma_w / T}$, and $0 \leq \sigma_z, \sigma_w \leq T$. The function $\mathrm{sgn}_\epsilon(\mathrm{x})$ is a smoothed version of the sign function $\mathrm{sgn}(\mathrm{x}) = \frac{\mathrm{x}}{|\mathrm{x}|}$; the transition from $\mathrm{sgn}_\epsilon(\mathrm{x}) = -1$ to $\mathrm{sgn}_\epsilon(\mathrm{x}) = 1$ occurs mostly over the interval $-\epsilon < x < \epsilon$. As $\epsilon \to 0$, $\mathrm{sgn}_\epsilon(\mathrm{x}) \to \mathrm{sgn}(\mathrm{x})$.

We would also like to combine exponentials which depend on the same coordinate. With the aid of (4.6), we see that

$$: e^{i\nu.X(z)} :: e^{i\mu \cdot X(z)} := (\epsilon z)^{\nu \cdot \mu} : e^{i(\nu + \mu) \cdot X(z)} : . \tag{4.14}$$

Applying (4.13) and (4.14), our typical term becomes

$$|typ\rangle = \epsilon^{\frac{1}{2} \sum_k \phi_k^2} z_1^{\frac{1}{2} \phi_1^2} \cdots z_n^{\frac{1}{2} \phi_n^2}$$

$$\exp\left(\frac{i\pi}{2} \sum_{j > k} \phi_j \cdot (1 - B) \cdot \phi_k \, \mathrm{sgn}_\epsilon (\sigma_j - \sigma_k) \right)$$

$$: e^{i\phi_1 \cdot X(z_1)} : \cdots : e^{i\phi_n \cdot X(z_n)} : |D'\rangle, \tag{4.15}$$

where we have introduced the convenient quantity $\phi_k = -(1 + M) \cdot \nu_k = -2(1 - B)^{-1} \cdot \nu_k$. Armed with equation (4.15), we can now address the question of conformal invariance. Requiring conformal invariance will lead us to a natural Kac-Moody algebra underlying our problem: the exponentials of X will be identified as Kac-Moody currents, and the vectors ϕ_j will be the root vectors. Once the algebraic structure is fleshed out, we will return to expression (4.4) and evaluate $|V\rangle$ explicitly in terms of Lie algebraic quantities.

4.2. Requirements for Conformal Invariance

We will now impose the following two constraints:

$$\phi_k^2 = 2 \qquad (4.16a)$$

$$\frac{1}{2}\phi_j \cdot (1 - B) \cdot \phi_k \in \mathbb{Z}. \qquad (4.16b)$$

With some work, we will see that (4.16a) and (4.16b) lead to conformal invariance of $|V\rangle$. We do not claim that these are necessary conditions for conformal invariance; indeed, calculations in [13] identify conformal theories that do not obey our constraints. Nevertheless, we choose to impose (4.16a) and (4.16b) because they lead to a large class of conformal theories where we can compute the boundary state exactly.

Equation (4.16a) ensures that the exponentials in (4.15) are weight one operators. It also ensures that (4.15) will have an overall factor of ϵ^n. Looking back at equation (4.4), we see that we will have precisely one factor of ϵ for each factor of V_ω. We can now dispose of ϵ: we rescale the bare couplings by $\frac{1}{\epsilon}$ and send $\epsilon \to 0$.

Equation (4.16b) allows us to replace the sgn function in (4.15) with a factor of unity. Evidence from the $N = 2$ version of this problem [13] suggests that conformal invariance holds when the sgn functions drop out. Setting sgn $= 1$ in (4.15) also disentangles the integrals over the z coordinates, allowing us to compute $|V\rangle$ very explicitly.

Noting that (4.16b) holds for all j and k, we can take the transpose and obtain

$$\frac{1}{2}\phi_j \cdot (1 + B) \cdot \phi_k \in \mathbb{Z}, \qquad (4.17)$$

which can be combined with (4.16b) to yield

$$\phi_j \cdot \phi_k \in \mathbb{Z}. \qquad (4.18)$$

Combined with (4.16a), (4.18) tells us that $\phi_j \cdot \phi_k \in \{0, \pm 1, \pm 2\}$.

These strong restrictions on ϕ_j immediately bring to mind the root systems for Lie algebras. The root vectors of any simply laced, semi-simple Lie algebra obey exactly the same constraints as the ϕ_j. Conversely, given a set of vectors ϕ_j subject to the constraints (4.16a) and (4.18), it can be proven that the set $\{\phi_j\}$ is a subset of the set of root vectors of some simply laced, semi-simple Lie algebra. These facts may be found in standard texts such as [16,17].

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Let us label the standard objects that arise for our Lie algebra: call the algebra g, let ψ_j denote the simple roots, and let $\mathcal{R}$ denote the set of roots. The root lattice $\Lambda_{\mathcal{R}}$ corresponds to the matrix consisting of the ψ_j as column vectors, and the weight lattice $\Lambda_{\mathcal{W}}$ is simply the dual of the root lattice. The Cartan matrix A is given by $A = \Lambda_{\mathcal{R}}^t \Lambda_{\mathcal{R}}$. Note that we use the same symbols for a lattice and the matrix of basis vectors for the lattice; for a summary of lattice conventions and facts, see the appendix.

So far, we have found that $\phi_j \in \mathcal{R}$, but we still have equation (4.16b) to deal with. It is convenient to recast (4.16b) in the form

$$\frac{1}{2}(1 - B)\Lambda_{\mathcal{R}} \subset \Lambda_{\mathcal{W}}. \tag{4.19}$$

In other words, we must have $\frac{1}{2}\Lambda_{\mathcal{W}}^{-1}(1 - B)\Lambda_{\mathcal{R}} = D$, where D is a matrix with integer entries. Solving for B, we find

$$B = 1 - 2\Lambda_{\mathcal{W}} D \Lambda_{\mathcal{W}}^t. \tag{4.20}$$

Requiring B to be antisymmetric tells us that D must satisfy the equation $D + D^t = A$. Noting that the Cartan matrix A is symmetric and has 2 in all the diagonal entries, we can solve for D:

$$D = 1 + \text{up}\,(A) + F, \tag{4.21}$$

where F is an arbitrary anti-symmetric matrix with integer entries, and $\text{up}\,(A)$ is the upper half of the matrix A (i.e., take A and set all entries on the diagonal and below to zero).

We now have the complete solution of our conformal constraints (4.16a) and (4.16b). First, we pick any simply laced, semi-simple Lie algebra g. The ϕ_j are root vectors of g, and the possible values for B are enumerated by (4.21) and (4.20). The original quantities ω_k which appear in our action are given by a linear transformation of the root vectors: $\{\omega_k\} = \frac{1}{2}(1 - B)\mathcal{R} \subset \Lambda_{\mathcal{W}}$. It is interesting to note that the gauge field is indispensable for realizing all algebras other than direct sums of $su(2)$: if we set $B = 0$, equation (4.20) tell us that $D = \frac{1}{2}\Lambda_{\mathcal{R}}^t \Lambda_{\mathcal{R}} = \frac{1}{2}A$. Since the Cartan matrix A has entries of -1 for all algebras other than $su(2)$, D will not be the required integer matrix unless g is a direct sum of $su(2)$ components.

4.3. Compactness Constraints

Let us consider the constraints due to compactification of X, namely equations (3.7) and (4.1). Recalling the definition $\phi_k = -(1 + M) \cdot \nu_k$, we can rewrite (4.1) in terms of the root lattice of g:

$$\frac{1}{\sqrt{2}}(1 - B)\Lambda_{\mathcal{R}} \subset \Gamma^*. \tag{4.22}$$

For convenience, we also redisplay (3.7):

$$\frac{1}{2}B\Gamma \subset \Gamma^*. \tag{4.23}$$

We need to address the following question: given our choice of Lie algebra g and our solution for B, what are the possible compactification lattices allowed by (4.22) and (4.23)? Clearly we can always choose X to be non-compact, regardless of B, because $\Gamma^* = \mathbb{R}^N$ in that case.

For compact X, equation (4.22) implies that

$$\frac{1}{\sqrt{2}}\Gamma^t(1 - B)\Lambda_{\mathcal{R}} = G, \tag{4.24}$$

where G is a matrix with integer entries. Using (4.20) to eliminate B, we can solve for Γ:

$$\Gamma = \frac{1}{\sqrt{2}}\Lambda_{\mathcal{R}}(D^{-1})^t G^t. \tag{4.25}$$

Equation (4.23) implies that $\frac{1}{2}\Gamma^t B\Gamma$ is a matrix with integer entries. Using (4.20) and (4.25) to eliminate B and Γ, we get the condition

$$\frac{1}{4}G\left(D^{-1} - (D^{-1})^t\right)G^t = \text{integer matrix}. \tag{4.26}$$

Any integer matrix G which satisfies (4.26) can be plugged into (4.25) to obtain a compactification lattice consistent with our constraints (4.22) and (4.23). Clearly there are many possible choices for G; for example, we can always use $G = 2\sqrt{k}D$ or $G = 2\sqrt{k}D^t$ with $k \in \mathbb{Z}$.

4.4. $|D'\rangle$ in terms of Kac-Moody representations

When we defined $|D'\rangle$ in equation (3.10), we did not make a definite choice for Υ, the lattice of momenta. Our definition of $|D'\rangle$ requires that $\frac{1}{\sqrt{2}}(1+B)\Gamma \subset \Upsilon$, and the conversion of right-movers into left-movers in equation (4.8) works on the assumption that $\omega_k \in \Upsilon$. We can satisfy these requirements by choosing $\Upsilon = \Lambda_\mathcal{W}$. To see this, first take the dual of equation (4.22):

$$\frac{1}{\sqrt{2}}(1+B)\Gamma \subset \Lambda_\mathcal{W}. \tag{4.27}$$

Next, recall that $\omega_k \in \frac{1}{2}(1-B)\mathcal{R} \subset \frac{1}{2}(1-B)\Lambda_\mathcal{R}$. Looking at equation (4.19), we see that $\omega_k \in \Lambda_\mathcal{W}$.

Once we set $\Upsilon = \Lambda_\mathcal{W}$, we can reinterpret $|D'\rangle$ in terms of Kac-Moody representations using the well-known vertex operator construction [18]. The level 1 Kac-Moody algebra $\hat{g}$ (i.e., the one corresponding to the Lie algebra g) can be written in terms of $\partial X(z)$ and exponentials of $X(z)$. The vertex operator construction requires us to restrict $p \in \Lambda_\mathcal{W}$, which is exactly the case with $|D'\rangle$. Each highest weight representation of $\hat{g}$ appears exactly once in the chiral Hilbert space of $X(z)$. These facts allow us to rewrite $|D'\rangle$ in the convenient form

$$|D'\rangle = 2^{-N/4}\sqrt{\det(1+B)\,\mathrm{vol}\,(\Gamma)} \sum_{\substack{\mu \in \mathcal{W} \\ s}} |\mu, s\rangle \overline{|\mu, s\rangle}, \tag{4.28}$$

where μ runs over the highest weights $\mathcal{W}$ of the level 1 highest weight representations of $\hat{g}$, and s labels the states of the representation μ.

4.5. Vertex Operators and Cocycles

We will now use the vertex operator construction [18] of the level 1 Kac-Moody algebra $\hat{g}$ to turn the exponentials in (4.15) into Kac-Moody currents. The algebra $\hat{g}$ has the commutation relations

$$[H_{ni}, H_{mj}] = n\,\delta_{ij}\delta_{n+m,0},$$

$$[H_n, E_m^\phi] = \phi E_{n+m}^\phi,$$

$$[E_n^\phi, E_m^{\phi'}] = \begin{cases} \epsilon(\phi, \phi')E_{n+m}^{\phi+\phi'}, & \text{if } \phi + \phi' \text{ is a root;} \\ \phi \cdot H_{n+m} + n\delta_{n+m,0}, & \text{if } \phi + \phi' = 0; \\ 0, & \text{otherwise.} \end{cases}$$

H_n are the Cartan operators, E_n^ϕ are the ladder operators, ϕ and ϕ' are root vectors, and $\epsilon(\phi, \phi')$ takes on the values ± 1. The Cartan operators are simply the modes of $\partial X(z)$: $H_{ni} = \alpha_{ni}$, and $H_{0i} = p_i$. In terms of currents, we have $H(z) = \sum_{n=-\infty}^{\infty} H_n z^{-n-1} = i\partial X(z)$. The ladder operators are constructed from the modes of vertex operators:

$$E^\phi(z) = \sum_{n=-\infty}^{\infty} E_n^\phi z^{-n-1} = : e^{i\phi \cdot X(z)} : \; c_\phi(p). \tag{4.30}$$

The chiral cocycle factor c_ϕ depends purely on the momentum operator p, which we explicitly indicate. Without the cocycles, the commutation relations for $\hat{g}$ do not work out properly. Following the treatment in [18] , we can explicitly construct the cocycles in the form

$$c_\phi(p) = \sum_{\phi' \in \Lambda_\mathcal{R}} \sum_{\mu \in \mathcal{W}} \epsilon(\phi, \phi') |p = \phi' + \mu\rangle\langle p = \phi' + \mu|. \tag{4.31}$$

Note that the sum over momenta is split into a sum over highest weight representations, labeled by μ, and the states in those representations, labeled by ϕ'. The coefficients $\epsilon(\phi, \phi')$ are the same ones that appear in the commutation relations for $\hat{g}$. Since we are free to change the definition of E_n^ϕ by a factor of -1, the function ϵ is not entirely fixed. Any function obeying the following constraints will do:

$$\epsilon(\alpha, \beta) \in \{1, -1\}, \tag{4.32a}$$

$$\epsilon(\alpha, \beta) = (-1)^{\alpha \cdot \beta} \epsilon(\beta, \alpha), \tag{4.32b}$$

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma), \tag{4.32c}$$

$$\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1, \tag{4.32d}$$

$$\epsilon(\alpha, -\alpha) = 1, \tag{4.32e}$$

for arbitrary $\alpha, \beta, \gamma \in \Lambda_\mathcal{R}$.

A good first guess is

$$\tilde{\epsilon}(\phi, \phi') = e^{i\frac{\pi}{2}\phi \cdot (1+B) \cdot \phi'}. \tag{4.33}$$

It is easily verified that $\tilde{\epsilon}$ satisfies all the conditions except for (4.32e). This may be remedied by defining

$$\eta_\phi \in \{1, -1\} \quad \text{chosen so that} \quad \eta_\phi \eta_{-\phi} = \tilde{\epsilon}(\phi, -\phi), \tag{4.34}$$

and setting

$$\epsilon(\phi, \phi') = \eta_\phi \, \eta_{\phi'} \, \eta_{\phi+\phi'} \, \tilde{\epsilon}(\phi, \phi'). \tag{4.35}$$

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For later convenience, we would like to fix the value of η_ϕ for the cases when ϕ is a root vector. Noting that $\eta_\phi \eta_{-\phi} = \tilde{\epsilon}(\phi, -\phi) = -1$ for all root vectors ϕ, we take

$$\eta_\phi = \begin{cases} 1, & \text{if } \phi \text{ is a positive root;} \\ -1, & \text{if } \phi \text{ is a negative root.} \end{cases} \tag{4.36}$$

Finally, we can use the fact that $c_\phi(p)^2 = 1$ to re-express (4.30):

$$: e^{i\phi \cdot X(z)} : = E^\phi(z) c_\phi(p). \tag{4.37}$$

4.6. Final Result for the Boundary State $|V\rangle$

Applying the results of the previous subsections, we can now write the typical term (4.15) in terms of Kac-Moody currents:

$$|typ\rangle = z_1 \cdots z_n \ e^{i\frac{\pi}{2}\sum_{j>k}\phi_j \cdot (1-B) \cdot \phi_k}$$
$$E^{\phi_1}(z_1) c_{\phi_1}(p) \cdots E^{\phi_n}(z_n) c_{\phi_n}(p) |D'\rangle.$$

Recall that we have eliminated ϵ by rescaling the bare couplings V_ω.

The ladder operator $E^\phi(z)$ carries momentum ϕ, so we can move the cocycles past the ladder operators, providing we shift the momenta:

$$|typ\rangle = z_1 \cdots z_n \ e^{i\frac{\pi}{2}\sum_{j>k}\phi_j \cdot (1-B) \cdot \phi_k}$$
$$E^{\phi_1}(z_1) \cdots E^{\phi_n}(z_n) c_{\phi_1}(p + \phi_2 + \cdots + \phi_n) c_{\phi_2}(p + \phi_3 + \cdots + \phi_n) \cdots c_{\phi_n}(p) |D'\rangle.$$

Shifting p simply means that

$$c_\phi(p+\beta) = e^{-iq\cdot\beta} c_\phi(p) e^{iq\cdot\beta} = \sum_{\phi'\in\Lambda_\mathcal{R}}\sum_{\mu\in\mathcal{W}} \epsilon(\phi, \phi'+\beta)|p = \phi'+\mu\rangle\langle p = \phi'+\mu|. \tag{4.40}$$

Applying (4.33), (4.34), and (4.35), we can simplify the cocycle product, obtaining

$$|typ\rangle = z_1 \cdots z_n \ E^{\phi_1}(z_1)\eta_{\phi_1} \cdots E^{\phi_n}(z_n)\eta_{\phi_n} \ U_{\sum_k \phi_k}(p)|D'\rangle. \tag{4.41}$$

All that remains of the cocycles are the numbers η_ϕ and the operator

$$U_\phi(p) = \sum_{\phi'\in\Lambda_\mathcal{R}}\sum_{\mu\in\mathcal{W}} \eta_{\phi'}\eta_{\phi+\phi'} e^{i\frac{\pi}{2}\phi \cdot (1+B) \cdot \phi'}|p = \phi'+\mu\rangle\langle p = \phi'+\mu|. \tag{4.42}$$

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Since $p|D'\rangle = \bar{p}|D'\rangle$, we can make the replacement $U_\phi(p)|D'\rangle = U_\phi(\bar{p})|D'\rangle$, allowing us to commute U to the left of the ladder operators:

$$|typ\rangle = \epsilon^n \ z_1 \cdots z_n \ U_{p-\bar{p}}(\bar{p}) \ E^{\phi_1}(z_1)\eta_{\phi_1} \cdots E^{\phi_n}(z_n)\eta_{\phi_n}|D'\rangle. \tag{4.43}$$

Plugging $|typ\rangle$ back into the series expansion (4.4), we see that we can perform the integrals over $z_1, \cdots z_n$ and resum the series into an exponential:

$$|V\rangle = P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R U_{p-\bar{p}}(\bar{p})$$

$$\exp\left(-T \sum_{\phi \in \mathcal{R}^+} \left(V_{-\frac{1}{2}(1-B)\phi}\eta_\phi E_0^\phi + \bar{V}_{-\frac{1}{2}(1-B)\phi}\eta_{-\phi} E_0^{-\phi}\right)\right)|D'\rangle. \tag{4.44}$$

Performing the contour integrals has left us with just the zero modes E_0^ϕ, where ϕ runs over the set of positive roots $\mathcal{R}^+$. Since the Kac-Moody fields are weight one, we have

$$[L_n, E_m^\phi] = -m E_{n+m}^\phi. \tag{4.45}$$

In particular, $[L_n, E_0^\phi] = 0$. Since we already know that $[L_n, S_B] = [L_n, R] = [L_n, p] = 0$, it follows that $(L_n - \bar{L}_n)|V\rangle = 0$. As promised, $|V\rangle$ is conformally invariant, regardless of the value of the couplings V_ω.

The only renormalization needed in our problem is a trivial rescaling of V_ω by the cutoff ϵ, which appeared when we normal ordered the exponentials in (4.10). Let us define the renormalized potential strengths in terms of the original (unrescaled) bare couplings: $\tilde{V}_\phi = i\epsilon T V_{-\frac{1}{2}(1-B)\phi}$ and $\tilde{V}_{-\phi} = -i\epsilon T \bar{V}_{-\frac{1}{2}(1-B)\phi}$ for positive roots ϕ. Our final result is

$$|V\rangle = P_{\sqrt{2}\Gamma}(p - \bar{p})S_B R U_{p-\bar{p}}(\bar{p}) \ \exp\left(i \sum_{\phi \in \mathcal{R}} \tilde{V}_\phi E_0^\phi\right)|D'\rangle. \tag{4.46}$$

This expression for $|V\rangle$ has a very appealing simplicity: we act on the basic state $|D'\rangle$ with a series of unitary operators, then project onto the space of allowed momenta. Each unitary operator has an obvious interpretation: $\exp\left(i\sum \tilde{V}_\phi E_0^\phi\right)$ is the Lie group element corresponding to the potential $V(X)$. S_B is the $SO(N)$ rotation due to the gauge field. The operator U, which only takes on the values ± 1, serves to keep track of what choice of signs we made in defining our Kac-Moody operators. Lastly, the reflection operator R allows us to switch between the Neumann boundary state and the more convenient Dirichlet boundary state.

5. Partition Functions

Now that we have obtained the boundary state created by the potential $V(X)$, we can compute the cylinder partition function with various boundary conditions on the opposite end. Due to the algebraic structure of the boundary state, all results can be expressed as traces over the Kac-Moody representation space. We will use the notation $\text{Tr}_{\hat{g}}(\cdots)$, meaning that we trace over all the states of each highest-weight representation of $\hat{g}$.

5.1. Neumann Case

Putting $|N\rangle$ at one end and $|V\rangle$ at the other, the partition function is

$$Z_{NV} = \langle N| e^{-\frac{2\pi}{T} l(L_0 + \bar{L}_0)} |V\rangle. \tag{5.1}$$

Setting $w = e^{-2\pi l/T}$ and applying (4.46), we get

$$
\begin{aligned}
Z_{NV} &= \langle N| w^{2L_0} P_{\sqrt{2}\Gamma}(p - \bar{p}) S_B R U_{p-\bar{p}}(\bar{p})\, e^{i \sum \tilde{V}_\phi E_0^\phi} |D'\rangle \\
&= \langle N| w^{2L_0} R P_{\frac{1}{\sqrt{2}}\Gamma}(p) U_{(M-1)\cdot p}(p) S_B e^{i \sum \tilde{V}_\phi E_0^\phi} |D'\rangle \\
&= 2^{-N/2} \sqrt{\det(1+B)}\, \text{vol}(\Gamma) \, \text{Tr}_{\hat{g}} \left(w^{2L_0} P_{\frac{1}{\sqrt{2}}\Gamma}(p) U_{(M-1)\cdot p}(p) S_B e^{i \sum \tilde{V}_\phi E_0^\phi} \right).
\end{aligned}
$$

In the special case where $g = su(2)$ and $B = 0$, our result simplifies to

$$Z_{NV} = 2^{-1/2}\, \text{vol}(\Gamma)\, \text{Tr}_{\hat{g}} \left(w^{2L_0} P_{\frac{1}{\sqrt{2}}\Gamma}(p) e^{i \sum \tilde{V}_\phi E_0^\phi} \right). \tag{5.3}$$

If we specialize further by taking the compactification lattice to be $\Gamma = \frac{1}{\sqrt{2}}\Lambda_{\mathcal{R}} = \mathbb{Z}$, we get a sum over the characters of $\hat{g}$:

$$Z_{NV} = 2^{-1/2}\, \text{Tr}_{\hat{g}} \left(w^{2L_0} e^{i \sum \tilde{V}_\phi E_0^\phi} \right). \tag{5.4}$$

5.2. Dirichlet Case

Putting $|D\rangle$ at one end and $|V\rangle$ at the other, the partition function is

$$
\begin{aligned}
Z_{DV} &= \langle D| w^{L_0 + \bar{L}_0} |V\rangle \\
&= \langle D| w^{2L_0} P_{\sqrt{2}\Gamma}(p - \bar{p}) S_B R U_{p-\bar{p}}(\bar{p})\, e^{i \sum \tilde{V}_\phi E_0^\phi} |D'\rangle \\
&= 2^{-N/2} \sqrt{\det(1+B)}\, \text{Tr}_{\hat{g}} \left(w^{2L_0} P_{\frac{1}{\sqrt{2}}\Gamma^*}(p) U_{-(1+M)p}(p) S_B R e^{i \sum \tilde{V}_\phi E_0^\phi} \right).
\end{aligned}
$$

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5.3. Gauge Field and Potential at Both Ends

For our final example, we will put the same gauge field state $|B\rangle$ and Lie group g at both ends, but we will allow the values of the V_ω to be different. Physically speaking, we are studying an open string whose ends feel periodic potentials $V^L(X)$ and $V^R(X)$ and a single uniform field B. It is important to note that the ends are *oppositely* charged as far as the gauge field is concerned. The partition function is

$$
Z_{VV} = \langle V^L | e^{-\frac{2\pi}{T} l (L_0 + \bar{L}_0)} | V^R \rangle
$$
$$
= \langle D' | e^{-i \sum \tilde{V}_\phi^L E_0^\phi} U_{p-\bar{p}}^\dagger(\bar{p}) R^\dagger S_B^\dagger P_{\sqrt{2}\Gamma}(p - \bar{p}) w^{2L_0} P_{\sqrt{2}\Gamma}(p - \bar{p}) S_B R U_{p-\bar{p}}(\bar{p}) \, e^{i \sum \tilde{V}_\phi^R E_0^\phi} | D' \rangle,
$$

again with $w = e^{-\frac{2\pi}{T} l}$. The rotations S_B and $S_B^\dagger$ for the two ends are conjugate, indicating that the string does indeed have zero net charge with respect to the gauge field. After all the conjugate operators cancel out, we get

$$
Z_{VV} = \langle D' | w^{2L_0} e^{-i \sum \tilde{V}_\phi^L E_0^\phi} \, P_{\sqrt{2}\Gamma}(M^t p + \bar{p}) \, e^{i \sum \tilde{V}_\phi^R E_0^\phi} | D' \rangle
$$
$$
= 2^{-N/2} \det(1 + B) \operatorname{vol}(\Gamma)
$$
$$
\sum_{\substack{\mu \in \mathcal{W} \\ s}} \langle \mu, s | w^{2L_0} e^{-i \sum \tilde{V}_\phi^L E_0^\phi} \, P_{\sqrt{2}\Gamma}(M^t p + p(\mu, s)) \, e^{i \sum \tilde{V}_\phi^R E_0^\phi} | \mu, s \rangle, \tag{5.7}
$$

where $p(u, s)$ is defined to be the momentum of the state $|\mu, s\rangle$, i.e., $p|\mu, s\rangle = p(\mu, s)|\mu, s\rangle$. The presence of $p(\mu, s)$ inside the projection operator prevents us from writing (5.7) as a trace. In the special case where $\tilde{V}_\phi^L = 0$, we can replace $p(\mu, s)$ with p, giving us

$$
Z_{BV} = 2^{-N/2} \det(1 + B) \operatorname{vol}(\Gamma) \operatorname*{Tr}_{\hat{g}} \left(w^{2L_0} P_{\frac{1}{\sqrt{2}}(1+B)\Gamma}(p) \, e^{i \sum \tilde{V}_\phi^R E_0^\phi} \right). \tag{5.8}
$$

6. Correlation Functions and the S-Matrix

The boundary state $|V\rangle$ provides a complete specification of the manner in which left-movers scatter into right-movers when they hit the boundary. The simplest way to extract the scattering data is to compute the correlation functions of ∂X and $\bar{\partial}\bar{X}$. Let us define the general correlation function on the cylinder

$$
F(z, \bar{\rho}) = \langle \partial X_{i_1}(z_1) \cdots \partial X_{i_n}(z_n) \bar{\partial}\bar{X}_{j_1}(\bar{\rho}_1) \cdots \bar{\partial}\bar{X}_{j_m}(\bar{\rho}_m) \rangle. \tag{6.1}
$$

In the boundary-state language, we have

$$
F(z, \bar{\rho}) = \frac{1}{Z_{CV}} \langle C | w^{2L_0} \partial X_{i_1}(z_1) \cdots \partial X_{i_n}(z_n) \bar{\partial}\bar{X}_{j_1}(\bar{\rho}_1) \cdots \bar{\partial}\bar{X}_{j_m}(\bar{\rho}_m) | V \rangle, \tag{6.2}
$$

where $w = e^{-2\pi l/T}$ and $< C|$ represents some arbitrary boundary condition at the other end of the cylinder.

Now recall equation (4.46) which shows how we can express $|V\rangle$ in terms of a projection operator and unitary operators acting on $|D'\rangle$. The projection operator commutes with ∂X and $\bar{\partial}\bar{X}$, so we may move it to the left. The unitary operators may also be moved to the left, conjugating the ∂X operators as they go by. This leaves the $\bar{\partial}\bar{X}$ operators next to the $|D'\rangle$ state, which can be used to convert them into ∂X operators. More precisely, the relation is $\bar{X}(\bar{z})|D'\rangle = -X(1/\bar{z})|D'\rangle$, so we have

$$F(z,\bar{\rho}) = \frac{1}{\bar{\rho}_1^2 \cdots \bar{\rho}_m^2 Z_{CV}} \langle C|w^{2L_0} P_{\sqrt{2}\Gamma}(p-\bar{p}) S_B R U_{p-\bar{p}}(\bar{p})\, e^{i\sum \tilde{V}_\phi E_0^\phi}$$
$$\partial \tilde{X}_{i_1}(z_1) \cdots \partial \tilde{X}_{i_n}(z_n) \partial X_{j_m}(1/\bar{\rho}_m) \cdots \partial X_{j_1}(1/\bar{\rho}_1)|D'\rangle. \tag{6.3}$$

The new operators $\tilde{X}$ are simply conjugated left-movers:

$$\partial \tilde{X}_i(z) = e^{-i\sum \tilde{V}_\phi E_0^\phi} R S_B^\dagger\, \partial X_i(z)\, S_B R e^{i\sum \tilde{V}_\phi E_0^\phi}. \tag{6.4}$$

Conjugation by S_B results in an $SO(N)$ rotation of the ∂X_i. Conjugation by R simply introduces a minus sign. Lastly, conjugation by the group element $e^{i\sum \tilde{V}_\phi E_0^\phi}$ results in a linear combination of the Cartan operators ∂X_i and ladder operators E^ϕ. Introducing the coefficients b_{jk}^V and $c_{j\phi}^V$ to describe the group rotation due to $V(X)$, we have:

$$\partial \tilde{X}_i(z) = -M_{ij}^t \left(b_{jk}^V \partial X_k(z) + c_{j\phi}^V E^\phi(z)\right). \tag{6.5}$$

For the sake of simplicity, we will now expand the cylinder into a half-plane by taking $T \to \infty$ and $l \to \infty$. Since $w \to 0$ as $l \to \infty$, only the vacuum amplitude in equation (6.3) survives. Furthermore, since $z = e^{2\pi(\tau+i\sigma)/T}$, $z \to 1$ and $\bar{z} \to 1$ as $T \to \infty$. It makes sense to change over to coordinates that are more convenient for the half-plane geometry, i.e., we switch to the coordinates $z = \tau + i\sigma$. In fact, we will go one step further and switch to the *open-string* picture on the half-plane: we use coordinates $z = \sigma + i\tau$, and the boundary is the imaginary axis $\sigma = 0$. The correlation functions are given by

$$\langle 0|\partial X_{i_1}(z_1) \cdots \partial X_{i_n}(z_n) \bar{\partial}\bar{X}_{j_1}(\bar{\rho}_1) \cdots \bar{\partial}\bar{X}_{j_m}(\bar{\rho}_m)|0\rangle_{open} =$$
$$\langle 0|\partial \tilde{X}_{i_1}(z_1) \cdots \partial \tilde{X}_{i_n}(z_n) \partial X_{j_m}(-\bar{\rho}_m) \cdots \partial X_{j_1}(-\bar{\rho}_1)|0\rangle_{free}. \tag{6.6}$$

The left-hand side of equation (6.6) is a correlation function for open-string fields propagating on the half-plane $\sigma > 0$ with the nontrivial action $S_{gauge} + S_{pot}$ acting at the spatial

boundary $\sigma = 0$. The right hand side of (6.6) is the same correlation function expressed entirely in terms of left-movers. Since the boundary interaction only affects correlations between different chiralities, the right-hand side of (6.6) can be considered a correlation function for chiral free fields on the full plane. Since $\tilde{X}$ is just a linear combination of ∂X and vertex operators, one may easily evaluate the right hand side of (6.6) with coherent state methods. Note that right-moving coordinates $\bar{\rho}_k$ are reflected to the image points $-\bar{\rho}_k$; singularities between left-movers and right-movers only occur when the coordinates coincide on the imaginary axis $\sigma = 0$.

The content of equation (6.6) is conceptually simple: we start with both chiralities in the half-plane (i.e., the left hand side of (6.6)), then we convert to a single chirality in the whole plane by reflecting the right-movers into image left-movers. The effect of the boundary interaction is summarized by the replacement $\partial X \rightarrow \partial \tilde{X}$. The operator $e^{-i \sum \tilde{V}_\phi E_0^\phi} R S_B^\dagger$ is the boundary S-matrix operator; the action of the S-matrix on our basic operators ∂X is summarized by (6.5).

7. Unitarity

Some recent papers [8,12] have commented on an apparent unitary violation when particles scatter from a boundary in conformal field theory. Although the particular models studied in [8] and [12] are different, the general features of the problem are the same: the S-matrix is explicitly computed, showing that certain ingoing states (i.e., combinations of left-movers) have less than unit probability of scattering into any combination of outgoing (i.e., right-moving) states. In some cases, the ingoing states seem to disappear entirely! The resolution of the paradox also takes the same general form in both cases: the Hilbert space of ingoing and outgoing states must be enlarged to include soliton states that were not originally included. In the enlarged Hilbert space, the S-matrix is perfectly unitary.

The model considered in this paper is a generalization of the $N = 1$ case studied in [12] , and so it is not surprising to find that the same unitarity question arises, with essentially the same solution. Let us begin by working in the Hilbert space built from the the operator ∂X. The scattering relation (6.5) shows us that outgoing states contain the ladder operators E^ϕ. Since the ladder operators carry momentum while the ∂X operators don't, it is clear that part of the outgoing state will be orthogonal to our Hilbert space built from ∂X. The solution is obvious: we need to study scattering in the Hilbert space which includes the soliton states created by the ladder operators.

One possible description of our enlarged Hilbert space is in terms of mutually orthogonal sectors: each sector is labeled by a momentum μ in the weight lattice $\Lambda_{\mathcal{W}}$. The sector with momentum μ is constructed by letting the modes of ∂X act on the μ-ground state $e^{iq \cdot \mu}|0\rangle$. This enlarged Hilbert space is nothing other than the direct sum of the highest-weight representations of the algebra $\hat{g}$, with each representation appearing exactly once. The ladder operators E^{ϕ} carry momentum equal to the root vector ϕ, so they create solitons interpolating between different μ-sectors.

Since the root lattice is a proper sublattice of the weight lattice, we do not actually have to use all the μ-sectors to describe scattering. The minimal choice is to use only the sectors labeled by root lattice momenta, in which case the Hilbert space is simply the highest-weight representation of $\hat{g}$ built from the vacuum state $|0\rangle$.

8. Conclusions

In this paper we have identified and solved a family of bosonic boundary conformal field theories with integer central charge. These theories exhibit a natural Kac-Moody current algebra structure, permitting us to compute the exact boundary state, partition function, and correlation functions. We find that conformal invariance is independent of the strength of the boundary potential, allowing us to wander over an entire manifold of fixed points by varying the couplings V_{ω}. Our results extend the $su(2)$-based calculations of [12,13] to the entire A-D-E series of simply laced semi-simple Kac-Moody algebras. It is worth noting that the gauge field part of the boundary interaction is absolutely vital for realizing all algebras other than $su(2)$.

These theories have at least two possible applications. In the context of string theory, our calculations identify a series of solutions of the classical open string equations: the spacetime fields which couple to the open string endpoints are a uniform abelian gauge field and a periodic potential for the tachyon.

Another application is the dissipative quantum mechanics of a charged particle moving in N dimensions, subject to a periodic potential and a magnetic field. Our results provide a complete description of the critical behaviour of such systems, although we have restricted ourselves to a special class of magnetic fields and potentials. Perturbative calculations [13] for $c = 2$ indicate that the Kac-Moody structure is destroyed at generic values of the magnetic field, so we do not expect any simple extension of our results to the case of arbitrary magnetic fields and potentials.

Finally, we offer the speculation that the type of models considered in our paper may have applications in edge current tunneling in the quantum Hall effect. Our reasoning is the same as that presented in [13], where a $c = 2$ version of our model was studied. In order to make any connection with the Hall effect, it would appear that we need to find an integrable deformation of our model. We hope to address the possibility of such an integrable model in future work.

Appendix . Lattices

A lattice on $\mathbb{R}^N$ is a set of vectors of the form

$$\Lambda = \Big\{ \sum_{i=1}^{N} n_i \lambda_i \ \Big| \ n_i \in \mathbb{Z} \Big\}, \tag{.1}$$

where the independent vectors $\lambda_1, \cdots \lambda_N$ are a basis for Λ. It is useful to define a matrix built out of the basis column vectors λ_i:

$$\Lambda_{matrix} = \big(\lambda_1 \lambda_2 \cdots \lambda_N \big). \tag{.2}$$

From now on, we will drop the 'matrix' subscript and use the single symbol Λ for both the lattice and the matrix; the context will make it clear which one is meant.

The unit cell of Λ is the set $\big\{ \sum_{i=1}^{N} x_i \lambda_i \ \big| \ 0 \leq x_i < 1 \big\}$. The lattice volume is defined to be the volume of the unit cell, given by

$$\mathrm{vol}\,(\Lambda) = \sqrt{|\det (\Lambda^t \Lambda)|} = |\det \Lambda|. \tag{.3}$$

The dual Λ^* of the lattice Λ is defined as

$$\Lambda^* = \Big\{ v \in \mathbb{R}^N \ \Big| \ v \cdot \lambda \in \mathbb{Z}, \ \forall \lambda \in \Lambda \Big\}, \tag{.4}$$

Given a basis λ_i for Λ, we can define a canonical basis λ_i^* for Λ^* such that $\lambda_i \cdot \lambda_j^* = \delta_{ij}$. In terms of matrices, we have the simple relation

$$\Lambda^* = \big(\Lambda^{-1} \big)^t = \big(\Lambda^t \big)^{-1}. \tag{.5}$$

The matrix representation makes it trivial to see that $\mathrm{vol}\,(\Lambda) = \mathrm{vol}\,(\Lambda^*)^{-1}$.

The matrix representation also makes it simple to express inclusion relations:

$$v \in \Lambda \iff \Lambda^{-1} \cdot v \in \mathbb{Z}^N$$

$$\text{and}$$

$$\Lambda \subset \Gamma \iff \Lambda^* \supset \Gamma^* \iff \Gamma^{-1}\Lambda \in \mathbb{Z}^{N \times N},$$

where $\mathbb{Z}^N$ is the set of vectors with N integer components and $\mathbb{Z}^{N \times N}$ is the set of N by N matrices with integer components.

As a notational convenience, it is useful to broaden the definition of a lattice to include any set of vectors closed under addition with integer coefficients. Under the broader definition, we can think of the set $\Lambda = \{0\}$ as a degenerate lattice. This allows us to write down expressions for compactified and uncompactified theories in the same notation.

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